Lecture 5 - Rotating Reference Frames

A Puzzle...

Recall that last time we considered various fun but bizarre physics phenomena. Here is yet another example of how simple physics can yield unexpected results.

Suppose that you stretch a slinky vertically to its full height above the ground, holding the slinky from the top. As shown in this video, when you release the slinky, the bottom end hangs in midair until the top end comes down and knocks into it. Explain why gravity does not seem to be applying to the bottom end of the slinky.

(1)

Solution

We are used to thinking of classical systems as instantly responding to stimuli. For example, if I place a meter stick between myself and a chair, and then push on the meter stick, the chair will instantly feel the push. But what if the stick is really long (say, a kilometer long)? It turns out that when the push the near end of the stick, the far end of the stick does not instantly "get the message." Instead, the atoms of stick near you push their neighbors, which in turn push their neighbors, and this signal propagates to the far end of the stick at the speed of sound (approximately 340 $\frac{\text{meters}}{\text{sec}}$). For a kilometer long stick, the far end of the stick will only start moving 3 seconds after you push the near end.

Now let's return to the slinky. When you release the top end of the slinky, the bottom end does not instantly know that the top has been released. In fact, all that the bottom end of the slinky knows is that gravity is acting upon it downwards, and tension from its neighboring chunk of slinky is acting upon it upwards, so that the bottom end of the slinky is in static equilibrium - *it does not matter that the top end of the slinky is no longer supported by your hand!* In the slinky system, because of the high tension before release, the information propagating down the slinky proceeds at nearly the same speed at the top of the slinky, so that the bottom of the slinky ends up hanging in midair until the top part of the slinky crashes into it. That said, you can check that the center of mass of the slinky falls downwards with acceleration g, so gravity is acting appropriately upon the slinky. □

Non-Inertial Reference Frames

The Importance of Non-Inertial Reference Frames

Why should we study non-inertial reference frames? Aside from their mathematical interest, we encounter non-inertial reference frames in our daily lives; linearly accelerating reference frames include riding on an elevator and bracing yourself on a subway while rotating reference frames are encountered inside of cars and carousels. Arguably, the most important non-inertial reference frame is the one that we are all in right now: on top of a rotating Earth. Understanding the forces that arise from a rotating reference frame enable us to model the surface of the Earth, which in turn helps us understand how g varies from one point to another on Earth.

In addition to its various applications, non-inertial reference frames provide us with beautiful mathematics, an excellent opportunity to work out our visualization skills, and an alternate route to problem solving that we can use to check our answers. When problems are amenable, I strongly recommend that you solve them from both a non-inertial and inertial reference frame.

Linearly Accelerating Reference Frames

One common point of confusion is whether "fictitious forces" arising in non-inertial reference frames are real. The answer is: it depends on your reference frame. Consider the following two scenarios.

- Scenario 1: You are sitting on a bench at a park. Across the street, you see a parkour kid get on top of a stationary car covered in ice and attempts to balance himself as his friend gets into the car and begins to drive. The parkour kid falls off the back of the car. Why did he fall off? Because his inertia kept him in place while the car accelerated out from under him. You smile and enjoy the rest of your day.
- Scenario 2: You are a parkour master, and after a difficult workout you and your friends want to go to an ice cream parlor one street away. Your friends inform you that if you are a true parkour master, you should be able to balance on top of the car, even though it had iced over the previous night and is essentially frictionless. You boldly accept the challenge, climb on top of the car, and brace yourself. As soon as the car starts moving, your legs begin to slowly slide backwards. You lean forward to brace yourself, but this invisible force mercilessly pulls you off the car, much to the amusement of some watchers on the other side of the street. Oh well, it is a quick jog to the ice cream parlor.

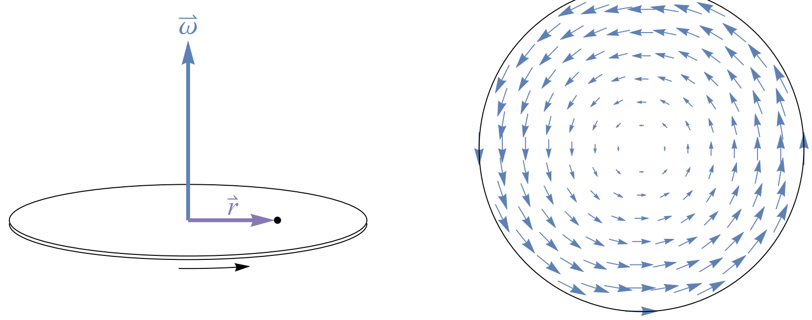
In Scenario 1, we are comfortable referring to your inertia as the reason why the parkour kid fell off the car, and in Scenario 2 the exact same thing happens, but from the perspective of Scenario 2 it feels as if there is an invisible force pulling you back. You have all felt this - for example, being pushed back into your seat when a car accelerates.

So is there a real force pushing you back? An outside observer (i.e. Scenario 1) would say that there is no real force, and that it is your inertia causing you to fall, and this observer would be right. An observer within the moving frame (i.e. Scenario 2) would say that there is an actual force causing them to slip backwards, and this observer would also be right. *Within the context of a particular reference frame, fictitious forces are real*. We call these fictitious forces because they depend on the reference frame (unlike gravity or a normal force which exist in any reference frame).

Although linearly accelerating reference frames are the most familiar, the rotating reference frames we turn to next come with a much more diverse array of fictitious forces.

Rotating Reference Frames: The Need for a Centrifugal Force

A carousel spins around its center at a constant angular velocity. The carousel can be completely described by the axis of rotation $\vec{\omega}$: the velocities of any point on the carousel are given by $\vec{v} = \vec{\omega} \times \vec{r}$. We will denote the magnitudes of $\vec{v}$ and $\vec{\omega}$ by $v \equiv |\vec{v}|$ and $\omega \equiv |\vec{\omega}|$.



Consider a person at a distance R from the center of the carousel who is *stationary relative to the carousel* (i.e. the person is sitting on top of the carousel as it spins). Recall that an objects undergoing uniform circular motion experience a radial acceleration $\frac{v^2}{R}$. The position $\vec{r} = R\,\hat{r}$ points radially outwards from $\vec{\omega}$ so that

$$v = |\vec{v}| = |\vec{\omega}\times\vec{r}| = |\vec{\omega}|\,|\vec{r}| = \omega\,R \tag{2}$$

Thus, the person's velocity would increase linearly with their radial distance from the center of a carousel. Using this equation, we can write the person's radial acceleration as

$$\frac{v^2}{R} = \omega^2\,R \quad \left(\begin{array}{c}\text{radial acceleration inwards}\\ \text{during uniform circular motion}\end{array}\right) \tag{3}$$

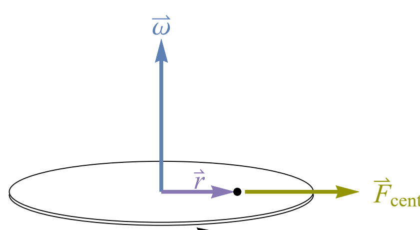
Example

Consider a person of mass m standing motionless on a carousel which rotates in the x-y plane with angular velocity $\vec{\omega} = \omega\,\hat{z}$. What is the centrifugal force felt by a person standing at a distance R from the center?

Solution

In the inertial reference frame, gravity will be balanced by the normal force while the friction at the person's feet will provide an inward radial force $m\,\omega^2\,R$.

If we now view the scenario from the rotating reference frame (where the carousel is *motionless*), gravity once again cancels the normal force, but there is no force to cancel the inward radial force $m\,\omega^2\,R$. Because the person must be standing still with respect to the carousel, there must be another force $\vec{F}_{\text{cent}}$ (soon to be called the *centrifugal force*) that points radially outwards with magnitude $\vec{F}_{\text{cent}} = m\,\omega^2\,R\,\hat{r}$.



Note that someone standing on the ground - in the inertial reference frame - will only see gravity, the normal force, and the friction force (and *not* the centrifugal force). However, *everyone can agree on the person's motion*: in the inertial reference frame of the ground the person is undergoing uniform circular motion with frequency ω at radius R while in the rotating reference frame the person stands still at radius R (which correspond to uniform circular motion with frequency ω at radius R). □

Centrifugal Force

So what is the formulation for the centrifugal force in a rotating reference frame? We will derive this force completely in the Advanced Section: Full Fictitious Forces section below. For now, we simply state the result:

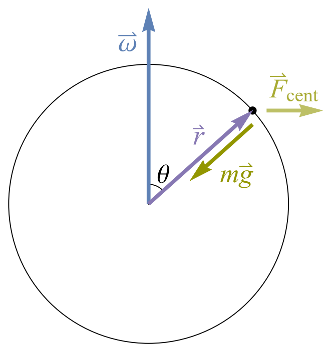
$$\vec{F}_{\text{cent}} = -m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}) \tag{4}$$

If we orient $\vec{\omega} = \omega\,\hat{z}$ and consider a radial vector $\vec{r} = R\,\hat{r}$, then (in cylindrical coordinates) $\vec{F}_{\text{cent}} = -m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -m\,\vec{\omega}\times(\omega\,R\,\hat{\theta}) = m\,\omega^2\,R\,\hat{r}$ which matches the form we found for the carousel problem above. A convenient fact to remember is that

$$\vec{F}_{\text{cent}} \textit{ always points directly away from } \vec{\omega} \tag{5}$$

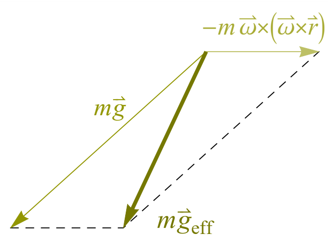
Example

Consider a person standing motionless on the earth, at a polar angle θ. She will feel a force due to gravity, $m\,\vec{g}$, directed toward the center of the earth. But in her rotating frame, she will also feel a centrifugal force, directed away from the rotation axis. The sum of these two forces (that is, what she thinks is gravity) will not point radially, unless she is at the equator or at a pole. Let us denote the sum of these forces as $m\,\vec{g}_{\text{eff}}$. Find $\vec{g}_{\text{eff}}$.



Solution

Adding together the gravitational force $m\,\vec{g}$ due to Earth's mass together with the centrifugal force $\vec{F}_{\text{cent}} = -m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$ yields $m\,\vec{g}_{\text{eff}} = m\,\vec{g} - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$ where the first term points towards the center of the Earth and the second term has magnitude $m\,\omega^2\,R\,\text{Sin}[\theta]$ where R is the radius of the Earth and points radially outwards from the $\vec{\omega}$ axis. $\vec{g}_{\text{eff}}$ can be found by vector addition



The magnitude of the correction term $m\,\omega^2\,R\,\text{Sin}[\theta]$ is small compared to g. Since the Earth revolves once per day,

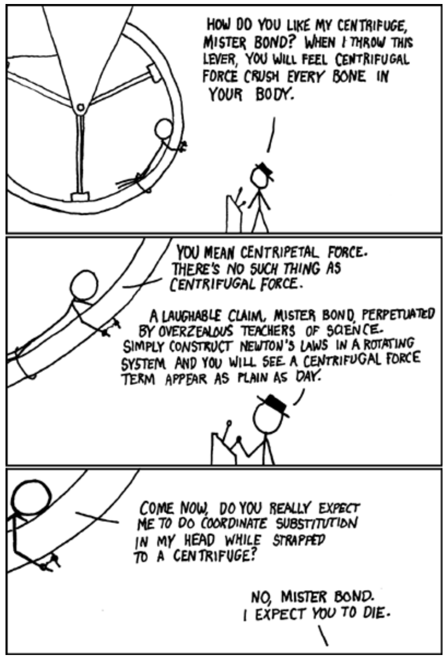
$$\omega \approx \frac{2\,\pi}{24\text{ hours}\left(3600\,\frac{\text{seconds}}{\text{hour}}\right)} = 7.27\times 10^{-5}\,\frac{1}{s} \tag{6}$$

Since the Earth's radius is roughly $R \approx 6.4\times 10^6\,m$, we find

$$R\,\omega^2 \approx 0.03\,\frac{m}{s^2} \tag{7}$$

which corresponds to a correction of about 0.3% to $\vec{g}_{\text{eff}}$ at the equator. Note that at the poles $\vec{g}_{\text{eff}} = \vec{g}$. □

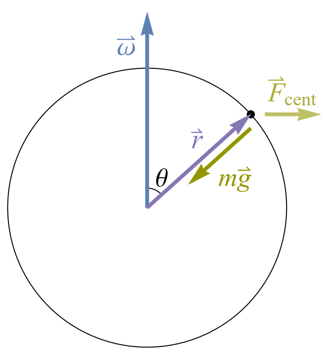
Note that what we generally call $\vec{g}$ (i.e. $9.8\,\frac{m}{s^2}$) is actually called $\vec{g}_{\text{eff}}$ in this problem, since it is $\vec{g}_{\text{eff}}$ that we actually measure (if you hang a plumb line down, it will point along $\vec{g}_{\text{eff}}$ and not $\vec{g}$).



Advanced Section: Shape of the Earth

Example

The earth bulges slightly at the equator, due to the centrifugal force in the earth's rotating frame. Approximate the height $h[\theta]$ of the Earth at every polar angle θ (where $\theta = 0$ corresponds to the North Pole).

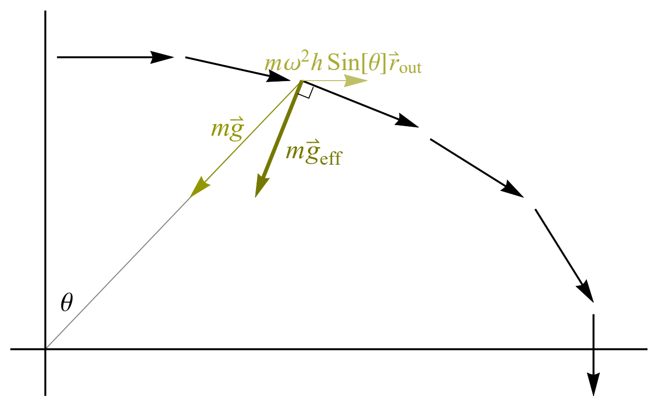


Solution

This problem is conceptually very easy, but setting up the differential equation is slightly tricky (and solving it requires some approximations plus a liberal use of *Mathematica*). Let's start with a game plan.

The surface of the Earth must be perpendicular to the gravitational force plus the centrifugal force $\vec{g}_{\text{eff}} = \vec{g} - \vec{\omega} \times (\vec{\omega} \times \vec{r})$. Thus, if we calculate the vector $\vec{g}_{\text{eff}}$ at each θ we can use that to guide what $\frac{dh}{d\theta}$ will be.

More precisely, consider the point shown in the diagram at the top-right of the sphere. Once we have found $\vec{g}_{\text{eff}}$ through vector addition, we can rotate $\vec{g}_{\text{eff}}$ by 90° counter-clockwise to find the *direction* (but not the magnitude) of the slope of the Earth at that point. We will then compute the proper magnitude of the slope and set it equal to $\frac{dh}{d\theta}$. Once we have $\frac{dh}{d\theta}$ for all θ, we can use it together with $h[0] = R$ and some approximations to find $h[\theta]$.



The centrifugal force has magnitude $m\,\omega^2\,h[\theta]\,\text{Sin}[\theta]$ pointing radially outwards from $\vec{\omega}$. This implies that there is a radial outwards component of $m\,\omega^2\,h[\theta]\,\text{Sin}[\theta]^2$ and a perpendicular component in the polar direction $\hat{\theta}$ of $m\,\omega^2\,h[\theta]\,\text{Sin}[\theta]\,\text{Cos}[\theta]$. Working in polar coordinates, the net force equals

$$m\,\vec{g}_{\text{eff}} = \left(-m\,g + m\,\omega^2\,h[\theta]\,\text{Sin}[\theta]^2\right)\hat{r} + \left(m\,\omega^2\,h[\theta]\,\text{Sin}[\theta]\,\text{Cos}[\theta]\right)\hat{\theta} \tag{8}$$

We rotate this vector 90° counter-clockwise to obtain the slope of the land at this value of θ

$$\text{slope} \propto \left(m\,\omega^2\,h[\theta]\,\text{Sin}[\theta]\,\text{Cos}[\theta]\right)\hat{r} + \left(m\,g - m\,\omega^2\,h[\theta]\,\text{Sin}[\theta]^2\right)\hat{\theta} \tag{9}$$

It is a subtle point that this is not the slope itself, but merely the direction of the slope. To calculate the actual slope, we want to calculate the amount that $h[\theta]$ increases (dh) when we increase θ to $\theta + d\theta$. To do this, we extend the slope line from the point $(h[\theta], \theta)$ until we reach the angle $\theta + d\theta$; and to do this the slope must cover the distance $h[\theta]\,d\theta$ (approximately) in the $\hat{\theta}$ direction. This will happen when we multiply the slope vector by some magnitude c satisfying

$$c\left(m\,g - m\,\omega^2\,h[\theta]\,\text{Sin}[\theta]^2\right) = h[\theta]\,d\theta \tag{10}$$

When the $\hat{\theta}$ term in the slope has traveled this distance, the $\hat{r}$ term in the slope will have increased $h[\theta]$ by

$$c\left(m\,\omega^2\,h[\theta]\,\text{Sin}[\theta]\,\text{Cos}[\theta]\right) = \frac{dh}{d\theta}\,d\theta \tag{11}$$

Dividing these two equations, we find

$$\frac{\omega^2\,h[\theta]^2\,\text{Sin}[\theta]\,\text{Cos}[\theta]}{g - \omega^2\,h[\theta]\,\text{Sin}[\theta]^2} = \frac{dh}{d\theta} \tag{12}$$

With the boundary condition $h[0] = R$, the problem is technically completely solved (up to integration, which can be done numerically). However, to simplify matters we make the approximation $g \gg \omega^2\,h[\theta]\,\text{Sin}[\theta]^2$ (which we saw above is definitely valid for the Earth) which yields

$$\frac{\omega^2}{g}\,h[\theta]^2\,\text{Sin}[\theta]\,\text{Cos}[\theta] = \frac{dh}{d\theta} \tag{13}$$

This differential equation can now be solved to obtain

$$h[\theta] = \frac{R}{1 - \frac{\omega^2 R}{2 g}\,\text{Sin}[\theta]^2} \tag{14}$$

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Clear[θ, R, ω]
DSolve[{h'[θ] == ω^2/g h[θ]^2 Sin[θ] Cos[θ], h[0] == R}, h[θ], θ]
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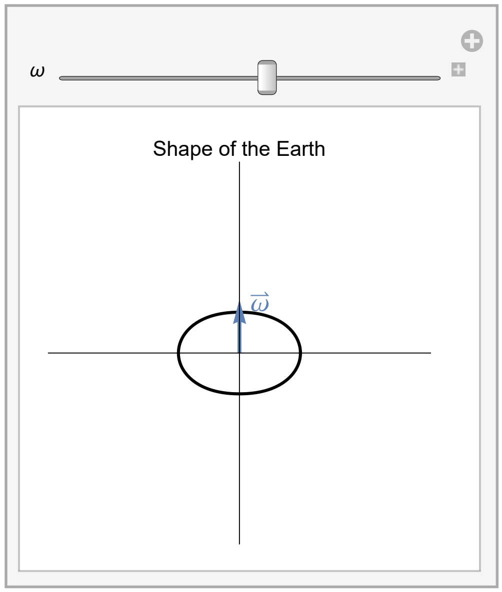
$$\left\{\left\{h[\theta] \to \frac{2\,g\,R}{2\,g - R\,\omega^2 + R\,\omega^2\,\text{Cos}[\theta]^2}\right\}\right\}$$

For small values of ω (such as the physical values for the Earth), $g \gg \omega^2 R$ and hence $h[\theta] \approx R$ implies that the Earth is basically a sphere. However, if we bump up ω (as shown in the plot below), the Earth bulge out along the equator compared to the poles. Why does this happen? Consider what happens if you ride on a carousel which spins faster and faster. As $\vec{\omega}$ increases, the centrifugal force outwards increases. No matter how tightly you hold on, if the carousel spins fast enough you will fly off of it. The exact same thing happens to the Earth. At the equator, there is a battle between the $m\,\vec{g}$ force inwards and the $\vec{F}_{\text{cent}} = -m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$ force outwards. The bigger $\vec{\omega}$ becomes, the bigger the centrifugal force, which will cause the Earth to bulge outwards. If ω is increased sufficiently, the planet will eventually fly apart.

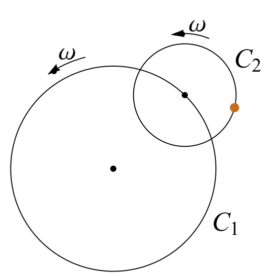


Note that in doing this computation, we have increased the total volume of the Earth. A way to avoid this is to choose our boundary condition $h[0]$ so that the total volume of the Earth is the same as the volume of a sphere; however, this will not qualitatively change our results. □

Visualizing a rotating reference frame

<u>Example</u>

Two circles in a plane, C_1 and C_2, each rotate with frequency ω relative to an inertia frame (alternatively, think of a carousel spinning on top of another carousel). The center of C_1 is fixed in an inertial frame, and the center of C_2 is fixed on C_1. A mass (orange point) is fixed on C_2. The position of the mass relative to the center of C_1 is $\vec{R}[t]$. Find the fictitious force felt by the mass.

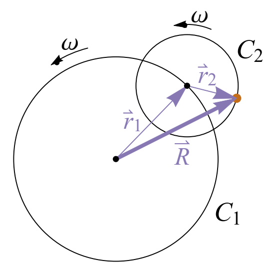


<u>Solution</u>

The fictitious force, $\vec{F}_f$, on the mass has an $\vec{F}_{\text{cent}}$ part (coming from the rotation of the mass in C_2) and an $\vec{F}_{\text{trans}}$ part (coming from the rotation of C_2 around C_1). So the fictitious force is

$$\vec{F}_f = m\,\omega^2\,\vec{r}_2 + \vec{F}_{\text{trans}} \tag{15}$$

where $\vec{r}_2$ is the position of the mass in the frame of C_2.



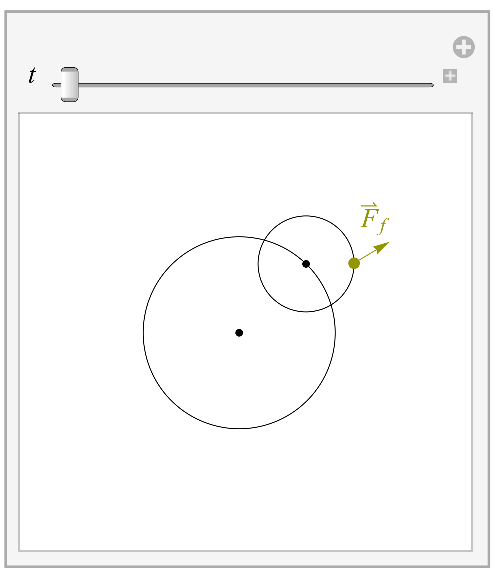
Furthermore, $\vec{F}_{\text{trans}}$, which arises from the acceleration of the center of C_2, is simply the centrifugal force felt by any point on C_1, and hence

$$\vec{F}_{\text{trans}} = m\,\omega^2\,\vec{r}_1 \tag{16}$$

where $\vec{r}_1$ is the position of the center of C_2 on C_1. Therefore,

$$\begin{aligned} \vec{F}_f &= m\,\omega^2\,\vec{r}_2 + m\,\omega^2\,\vec{r}_1 \\ &= m\,\omega^2(\vec{r}_1 + \vec{r}_2) \\ &= m\,\omega^2\,\vec{R}[t] \end{aligned} \tag{17}$$

Such a simple answer demands a simple explanation! To get a feel for why the result only depends on $\vec{R}[t]$, let us look at the motion of the point.



Notice that as time progresses, the system looks exactly the same as at $t = 0$ except that everything has been rotated by and angle $\omega\,t$. Indeed, a little thought shows that this is exactly what happens. In other words, we can imagine that all of the circles have been glued together, and it is easy to see that such a setup would indeed yield the same ω for all of our circles! With this picture in mind, it is clear that the mass is always a distance $\vec{R}[t] = \vec{R}[0]$ away from the origin and the net forces on it must be the centrifugal force $m\,\omega^2\,\vec{R}[t]$. □

Advanced Section: Full Fictitious Forces

Implications of Non-Inertial Forces

Example

You are floating high up in a balloon, at rest with respect to the earth. Give three quasi-reasonable definitions for which point on the ground is right "below" you.

Solution

1) The point that lies along the line between you and the center of the earth

2) The point where a hanging plumb bob rests

3) The point where a dropped object hits the ground

The first definition points towards $\vec{g}$ while the second towards $\vec{g}_{\text{eff}}$; the difference is caused by the centrifugal force. The third definition differs from the second because of the Coriolis force (the velocity of the falling object will cause a deflection in the objects path). □

Mathematica Initialization